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OPTIMAL DESIGN OF INTERNAL RING SUPPORTS FOR VIBRATING CIRCULAR PLATES

F. S. CHOU
Department of Decision Sciences, The National University of Singapore, Kent Ridge, Singapore 119260

C. M. WANG
Department of Civil Engineering, The National University of Singapore, Kent Ridge, Singapore 119260

G. D. CHENG
Dalian University of Technology, Dalian 116023, *People's Republic of China*

AND

N. OLHOFF
Institute of Mechanical Engineering, Aalborg University, Pontoppidanstraede 101, *DK* 9220 *Aalborg East, Denmark*

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This paper presents the optimality conditions for optimum locations of internal ring supports in circular plates for maximum frequency of axisymmetric vibration of a specified order. The optimality conditions require both the reaction forces and moments at the internal supports to vanish for optimal solutions. This means that the internal ring supports should be placed at the nodal rings of the appropriate higher-order vibration mode of a corresponding unsupported circular plate for maximum effectiveness in raising the vibration frequency of the plate. Some design examples are given to verify the derived optimality conditions.

1. INTRODUCTION

The positions and shape of the structural supports/restraints are usually specified by the architects or designers at the outset. The engineers then proceed to design the structure or machine to achieve the prescribed performance criteria. The design variables may consist of structural dimensions, shape parameters, topological parameters and material properties such as fibre orientations in laminated plates. As it is desirable to obtain an economical design, these variables are somewhat optimized, by the engineers, subject to appropriate constraints. One way of getting even better designs is to make a paradigm shift from fixing

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the supports/restraints at the outset to allowing engineers the freedom of designing more optimal supports/restraints, i.e. with respect to their locations, stiffnesses and shape.

In this study, we consider the optimal design of internal ring supports in circular plates for maximum specified order of frequency of axisymmetric vibration. Interestingly, the optimal locations of a given number of rigid internal supports which maximizes the fundamental frequency of transverse virbation of a string or a beam, or the elastic buckling load of an axially compressed column, are in fact found to be at the nodal points of an appropriate higher-order vibration or buckling mode of the original structural element. The proofs for this fact may be obtained from papers by Courant and Hilbert [1], Olhoff [2], Rozvany and Mroz [3], Olhoff and Taylor [4].

Even more remarkable is that the optimal design of a tapered beam with respect to its nth eigenfrequency ω_n is coincidently the optimal design with respect to the fundamental frequency ω_1 of a multiply supported beam, where the positions of $(n-1)$ simple supports between the beam end points (assumed to be additional design variables) as found to be at the nodes of the former design. This theorem has been proven by Olhoff [2]. The optimization of bracing and internal support locations for beams against lateral buckling was also investigated by Wang *et al*. [5].

Studies on the use of flexible supports have also been carried out by Szelag and Mroz [6, 7], Mroz and Lekszycki [8], Garstecki and Mroz [9], Akesson and Olhoff [10] and Olhoff and Akesson [11]. It was discovered that so long as the stiffnesses of the flexible supports take on sufficiently finite (or critical stiffness) values that will ensure a zero or near zero displacement at their locations, the nodal positions of an unsupported structural element will still give the optimal loations for such flexible supports. Obviously, when the stiffnesses fall below these critical values, then the optimal locations of the flexible supports do not coincide with the nodal positions. Moreover, the fundamental frequency of the internally supported structural element will decrease from its maximum value.

This paper extends the optimality conditions for optimal support locations for beams and columns to cover circular plates undergoing axisymmetric vibration. By adopting a more refined plate theory of Mindlin [12], the effects of transverse shear deformation and rotary inertia (which are significant in thick plates) on the optimal locations can also be examined. For verification and exemplification of the optimality conditions derived herein, some application oriented examples are solved. In these examples, the thickness distribution of the tapered circular plates is assumed to take the linearly segmented form.

2. PROBLEM DEFINITION

Consider an axisymmetric vibrating, circular plate with a thickness distribution $h(r)$, volume V, radius R, modulus of elasticity E, Poisson's ratio v and shear modulus $G = E/[2(1+v)]$. The plate is either simply supported or clamped at its periphery $r = R$ and further supported by a number M of internal ring supports at locations $r = e_1, e_2, \ldots, e_M$. The optimization problem is to determine the optimal locations e_i of the ring supports so as to maximize the plate axisymmetric vibration frequency of a specified order.

3. GOVERNING EQUATIONS AND OPTIMALITY CONDITIONS FOR SUPPORT LOCATIONS

Based on the Mindlin plate theory for harmonic, axisymmetric vibration, the Rayleigh quotient for the plate circular frequency ω is given by [12, 13]

$$\omega^2 = \operatorname*{Min}_{w,\psi}\left(\frac{U}{T}\right), \qquad (w, \psi)\in S_a, \tag{1}$$

where w is the transverse displacement, ψ the rotation of the normal of the cross-section, S_a is the set of kinematically admissible displacement fields which consists of all displacements satisfying the necessary continuity conditions and kinematic boundary conditions and

$$U = \pi \int_0^R \left\{D\left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 + \left(\frac{\psi}{r}\right)^2 + 2\nu\frac{\psi}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right] + \kappa^2 Gh\left(\psi + \frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right\} r\,\mathrm{d}r, \tag{2a}$$

$$T = \pi \int_0^R \mu h\left\{w^2 + \frac{h^2\psi^2}{12}\right\} r\,\mathrm{d}r, \tag{2b}$$

where r is the radial co-ordinate measured from the plate center, $D(r) = Eh^3(r)/[12(1-\nu^2)]$ is the flexural rigidity of the plate, μ the mass density of the plate, κ^2 the Mindlin shear correction factor which will be assumed to be 5/6 for the present study.

The minimization of the Rayleigh quotient with respect to the displacement fields w, ψ yields the governing differential equations and the boundary conditions. By taking the stationarity condition of ω^2 with respect to the deflection w, one obtains

$$\delta_w \omega^2 = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}[rQ_r] + \omega^2 \mu h r w = 0, \tag{3}$$

and the boundary conditions given by

$$Q_r = 0 \quad \text{at} \quad r = 0, \qquad w = 0 \quad \text{at} \quad r = R, \tag{4a, b}$$

with the transverse shear force Q_r given by

$$Q_r = \kappa^2 Gh\left(\psi + \frac{\mathrm{d}w}{\mathrm{d}r}\right). \tag{5}$$

The stationarity condition of ω^2 with respect to ψ gives

$$\delta_\psi \omega^2 = 0$$

$$\Rightarrow -\frac{\mathrm{d}}{\mathrm{d}r}(rM_r) + M_\theta + rQ_r - \omega^2 \mu \frac{h^3}{12}\psi r = 0, \tag{6}$$

where the radial bending moment M_r and the circumferential bending moment M_θ are given by

$$M_r = D\left(\frac{\mathrm{d}\psi}{\mathrm{d}r} + \frac{v}{r}\psi\right), \qquad M_\theta = D\left(v\frac{\mathrm{d}\psi}{\mathrm{d}r} + \frac{1}{r}\psi\right), \tag{7, 8}$$

and the corresponding boundary conditions at $r = 0$ and $r = R$ are given by

$$\psi = 0 \qquad \text{at} \quad r = 0, \tag{9a}$$

$$M_r = 0 \qquad \text{at} \quad r = R \quad \text{for a simply supported plate}, \tag{9b}$$

$$\psi = 0 \qquad \text{at} \quad r = R \quad \text{for a clamped plate}. \tag{9c}$$

Before proceeding further, the continuity conditions of various quantities should be noted. The displacement w and the rotation ψ must be continuous but their derivatives $\mathrm{d}w/\mathrm{d}r$ and $\mathrm{d}\psi/\mathrm{d}r$ need not be continuous over the support. Owing to the equilibrium condition, the radial bending moment M_r and transverse shear force Q_r are continuous between any two neighbouring supports. At the support position, the shear force and the radial moment distributions may be discontinuous. The jumps in the shear force value and the radial moment value are equal to the reaction force P_{wi} and the reaction moment $P_{\psi i}$ provided by the support, respectively.

Using the transversality condition for internal subdomain boundaries (e.g., reference [14]), the optimality condition for the locations of the internal ring supports is given by

$$(F - w'F_{w'} - \psi' F_{\psi'})|_{e_i^-} = (F - w'F_{w'} - \psi' F_{\psi'})|_{e_i^+}, \qquad i = 1, 2, \ldots, M, \tag{10}$$

where $\int F\,\mathrm{d}r = U - \omega^2 T$, $(\bullet)' = \mathrm{d}(\bullet)/\mathrm{d}r$, $F_{(\bullet)} = \mathrm{d}F/d(\bullet)$ and e_i^+, e_i^- are the right and left sides, respectively of the point $r = e_i$. In view of the continuity condition of w and ψ across the internal support, equation (10) may be written as

$$\left\{-Dr\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 - \kappa^2 Ghr\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right\}\Bigg|_{e_i^-} = \left\{-Dr\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)^2 - \kappa^2 Ghr\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right\}\Bigg|_{e_i^+},$$

$$i = 1, 2, \ldots, M. \tag{11}$$

For unspecified ψ at the internal support, one has the transversality condition:

$$(F_{\psi'})|_{e_i^-} - (F_{\psi'})|_{e_i^+} = 0, \qquad i = 1, 2, \ldots, M. \tag{12}$$

As ψ is continuous over the internal support, equation (12) yields the optimality condition

$$\left(\frac{d\psi}{dr}\right)\bigg|_{e_i^-} = \left(\frac{d\psi}{dr}\right)\bigg|_{e_i^+}, \qquad i = 1, 2, \ldots, M, \tag{13}$$

and when substituted into equation (11), one can also conclude that

$$\left(\frac{dw}{dr}\right)\bigg|_{e_i^-} = \left(\frac{dw}{dr}\right)\bigg|_{e_i^+}, \qquad i = 1, 2, \ldots, M. \tag{14}$$

Thus, the continuity of w, ψ, dw/dr, $d\psi/dr$ at the optimal support location implies that

$$(Q_r)|_{e_i^+} - (Q_r)|_{e_i^-} = P_{wi}|_{e_i} = 0, \qquad i = 1, 2, \ldots, M, \tag{15}$$

$$M_r|_{e_i^+} - M_r|_{e_i^-} = P_{\psi i}|_{e_i} = 0, \qquad i = 1, 2, \ldots, M. \tag{16}$$

In summary, the conditions given by equations (15) and (16) for the optimal location of an internal ring support require that both the reaction force and the reaction moment should vanish at the support. This means that the optimal locations of M internal ring supports may be determined by seeking the M nodal rings (circular contours with $w = 0$) of the $(M + 1)$ axisymmetric vibration mode of an internally unsupported plate.

4. DESIGN EXAMPLES

4.1. EXAMPLE (1): UNIFORM THICKNESS PLATES WITH RING SUPPORTS

For the first example, simply supported and clamped circular plates of two different thickness values $h_o/R = 0{\cdot}001$ and $0{\cdot}3$ were analysed using the Rayleigh–Ritz method (see paper by Chou and Wang [13]). The first thickness value depicts a very thin plate and this would be equivalent to adopting the classical thin plate theory where the effect of transverse shear deformation is neglected. Thus, the comparison of the two plate results would reveal the effect of the transverse shear deformation. In all calculations, the Poisson ratio $\nu = 0{\cdot}3$ and the shear correction factor $\kappa^2 = 5/6$ were adopted.

Table 1 compares the maximum fundamental frequency parameters $\lambda_1 = \omega_1^2 \mu h_o R^4/D_o$ of plates with M optimally located, internal supports to the $(M + 1)$ natural frequency parameters $\breve{\lambda}_{m+1} = \breve{\omega}_{M+1}^2 \mu h_o R^4/D_o$ of plates without internal support. An exact matching of the frequency values was observed and also the optimal support locations e_i were found at the nodal locations $\breve{e}_i$ of the corresponding plates without internal support. The numerical results thus verify the foregoing optimality conditions for support locations, i.e., the supports should be placed at the nodal ring positions of the appropriate higher vibration mode.

For clamped plates, it can be observed from the vibration mode shapes in Figure 1(a) and the results in Table 1 that the effects of transverse shear deformation and rotary inertia tend to shift the ring supports (or nodal positions)

TABLE 1

Comparison between fundamental frequency parameters and optimal support locations of plates with internal ring supports and the appropriate natural frequency parameters and nodal ring positions of corresponding plates without any internal ring support

		Simply supported edge		Clamped edge	
h_o/R	M	Plate with M internal support	Plate without internal support	Plate with M internal support	Plate without internal support
0·001	1	$\lambda_1 = 883{\cdot}3$ $e_1/R = 0{\cdot}442$	$\check{\lambda}_2 = 883{\cdot}3$ $\check{e}_1/R = 0{\cdot}442$	$\lambda_1 = 1582$ $e_1/R = 0{\cdot}379$	$\check{\lambda}_2 = 1852$ $\check{e}_1/R = 0{\cdot}379$
0·3	1	$\lambda_1 = 468{\cdot}5$ $e_1/R = 0{\cdot}442$	$\check{\lambda}_2 = 468{\cdot}5$ $\check{e}_1/R = 0{\cdot}442$	$\lambda_1 = 611{\cdot}2$ $e_1/R = 0{\cdot}396$	$\check{\lambda}_2 = 611{\cdot}2$ $\check{e}_1/R = 0{\cdot}396$
0·001	2	$\lambda_1 = 5490$ $e_1/R = 0{\cdot}279$ $e_2/R = 0{\cdot}641$	$\check{\lambda}_3 = 5490$ $\check{e}_1/R = 0{\cdot}279$ $\check{e}_2/R = 0{\cdot}641$	$\lambda_1 = 7939$ $e_1/R = 0{\cdot}255$ $e_2/R = 0{\cdot}583$	$\check{\lambda}_3 = 7939$ $\check{e}_1/R = 0{\cdot}255$ $\check{e}_2/R = 0{\cdot}583$
0·3	2	$\lambda_1 = 1768$ $e_1/R = 0{\cdot}279$ $e_2/R = 0{\cdot}641$	$\check{\lambda}_3 = 1768$ $\check{e}_1/R = 0{\cdot}279$ $\check{e}_2/R = 0{\cdot}641$	$\lambda_1 = 1914$ $e_1/R = 0{\cdot}272$ $e_2/R = 0{\cdot}616$	$\check{\lambda}_3 = 1914$ $\check{e}_1/R = 0{\cdot}272$ $\check{e}_2/R = 0{\cdot}616$

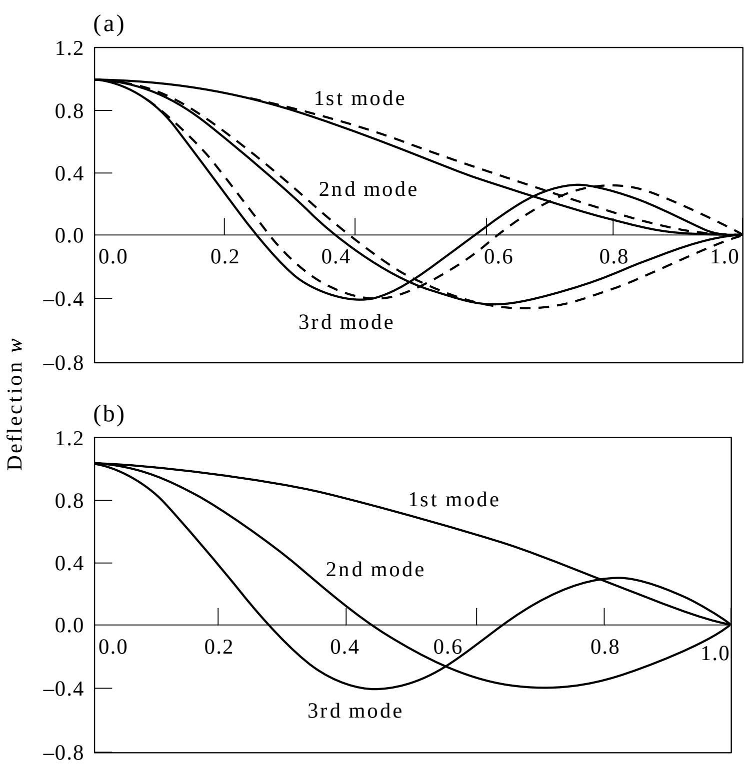

Figure 1. First three vibration mode shapes of uniform thickness plates without internal ring supports for (a) clamped edges and (b) simply supported edges. —, $h/R = 0{\cdot}001$; - - - , $h/R = 0.30$.

slightly towards the clamped edge. Interestingly, these effects have negligible influence in the case of simply supported plates with uniform thickness, as shown by the vibration mode shapes in Figure 1(b). This means that the optimal locations of internal ring supports in circular, thick plates with simply supported edges are found at the nodal rings of a correspondingly thin, simply supported, circular plates. The nodal radii $\check{e}$ for the axisymmetric vibration of thin plates may be determined from the following equation [15]

$$I_0(k\check{e})J_0(kR) - I_0(kR)J_0(k\check{e}) = 0, \tag{17}$$

where J_0 and I_0 are the zeroth order Bessel functions of the first kind and modified first kind, respectively, and the frequency parameter $k^4 = \bar{\omega}^2\rho h/D$ for the thin, simply supported circular plate is to be computed from the frequency equation

$$\frac{J_1(kR)}{J_0(kR)} + \frac{I_1(kR)}{I_0(kR)} = \frac{2kR}{1-\nu}\,. \tag{18}$$

Interestingly, Wang [16] has shown that the Mindlin plate vibration frequencies ω may be deduced quite accurately from the corresponding classical thin plates vibration frequencies $\bar{\omega}$ using the following frequencies relationship:

$$\omega^2 = \frac{6\kappa^2 G}{\mu h^2}\left\{\left[1 + \frac{1}{12}\,\bar{\omega}h^2\sqrt{\frac{\mu h}{D}}\left(1 + \frac{2}{\kappa^2(1-\nu)}\right)\right]\right.$$
$$\left. - \sqrt{\left[1 + \frac{1}{12}\,\bar{\omega}h^2\sqrt{\frac{\mu h}{D}}\left(1 + \frac{2}{\kappa^2(1-\nu)}\right)\right]^2 - \frac{\mu h^2}{3\kappa^2 G}\,\bar{\omega}^2}\right\}. \tag{19}$$

Note that the same relationship has been proven to be exact for the case of general polygonal plates with simply supported edges [17].

4.2. EXAMPLE (2): PIECEWISE LINEARLY SEGMENTED PLATES WITH RING SUPPORTS

For this next illustrative design example, simply supported and clamped, continuous linearly segmented plates with N equal length (radial) segments are considered. As before, to examine the effects of transverse shear deformation and rotary inertia, two thickness parameters $h_o/R = 0{\cdot}001$ and $0{\cdot}3$ are considered, where h_o is equal to the thickness of a reference, uniform thickness plate having the same volume as the segmented plate. Here we optimize the plate thickness parameters $\tau = h/h_o$ at the segment boundaries as well as the locations of the internal ring supports for maximum fundamental frequency parameter $\lambda_1 = \omega_1^2\mu h_o R^4/D_o$. The details of the optimization method may be obtained from the paper by Chou and Wang [13]. The optimal solutions are tabulated in Tables 2 and 3 for one and two ring support cases and one to two segments. Next, the segmented plate without any ring supports was optimized for maximum second frequency parameter $\check{\lambda}_2 = \check{\omega}_2^2\mu h_o R^4/D_o$ and also for maximum third frequency parameter $\check{\lambda}_3 = \check{\omega}_3^2\mu h_o R^4/D_o$ of vibration and the results are shown in Tables 2 and 3. It may be observed from the tables that the nodal ring positions are almost

TABLE 2

Comparison between the optimal solutions of ($N = 1$) *segmented plates with internal ring supports and the corresponding plates without any ring support*

		Simply supported edge		Clamped edge	
h_o/R	M	Plate with M internal support	Plate without internal support	Plate with M internal support	Plate without internal support
0·001	1	$\lambda_1 = 1089$ $e_1/R = 0{\cdot}475$ $\tau(0) = 2{\cdot}141$ $\tau(1) = 0{\cdot}430$	$\breve{\lambda}_2 = 1089$ $\breve{e}_1/R = 0{\cdot}475$ $\tau(0) = 2{\cdot}140$ $\tau(1) = 0{\cdot}430$	$\lambda_1 = 1679$ $e_1/R = 0{\cdot}410$ $\tau(0) = 1{\cdot}591$ $\tau(1) = 0{\cdot}705$	$\breve{\lambda}_2 = 1679$ $\breve{e}_1/R = 0{\cdot}411$ $\tau(0) = 1{\cdot}591$ $\tau(1) = 0{\cdot}705$
0·3	1	$\lambda_1 = 509{\cdot}4$ $e_1/R = 0{\cdot}451$ $\tau(0) = 1{\cdot}979$ $\tau(1) = 0{\cdot}511$	$\breve{\lambda}_2 = 509{\cdot}4$ $e_1/R = 0{\cdot}451$ $\tau(0) = 1{\cdot}979$ $\tau(1) = 0{\cdot}511$	$\lambda_1 = 619{\cdot}8$ $e_1/R = 0{\cdot}409$ $\tau(0) = 1{\cdot}445$ $\tau(1) = 0{\cdot}777$	$\breve{\lambda}_2 = 619{\cdot}8$ $\breve{e}_1/R = 0{\cdot}409$ $\tau(0) = 1{\cdot}446$ $\tau(1) = 0{\cdot}777$
0·001	2	$\lambda_1 = 6855$ $e_1/R = 0{\cdot}328$ $e_2/R = 0{\cdot}701$ $\tau(0) = 2{\cdot}092$ $\tau(1) = 0{\cdot}454$	$\breve{\lambda}_2 = 6855$ $\breve{e}_1/R = 0{\cdot}327$ $\breve{e}_2/R = 0{\cdot}700$ $\tau(0) = 2{\cdot}086$ $\tau(1) = 0{\cdot}457$	$\lambda_3 = 9153$ $e_1/R = 0{\cdot}297$ $e_2/R = 0{\cdot}645$ $\tau(0) = 1{\cdot}843$ $\tau(1) = 0{\cdot}576$	$\breve{\lambda}_3 = 9154$ $\breve{e}_1/R = 0{\cdot}296$ $\breve{e}_2/R = 0{\cdot}644$ $\tau(0) = 1{\cdot}837$ $\tau(1) = 0{\cdot}582$
0·3	2	$\lambda_1 = 1856$ $e_1/R = 0{\cdot}293$ $e_2/R = 0{\cdot}660$ $\tau(0) = 1{\cdot}822$ $\tau(1) = 0{\cdot}589$	$\breve{\lambda}_3 = 1856$ $\breve{e}_1/R = 0{\cdot}293$ $\breve{e}_2/R = 0{\cdot}660$ $\tau(0) = 1{\cdot}822$ $\tau(1) = 0{\cdot}589$	$\lambda_1 = 1994$ $e_1/R = 0{\cdot}286$ $e_2/R = 0{\cdot}639$ $\tau(0) = 1{\cdot}784$ $\tau(1) = 0{\cdot}608$	$\breve{\lambda}_3 = 1994$ $\breve{e}_1/R = 0{\cdot}286$ $\breve{e}_2/R = 0{\cdot}638$ $\tau(0) = 1{\cdot}775$ $\tau(1) = 0{\cdot}612$

in total agreement with the optimal locations of the ring supports and the same applies for the optimal thickness parameters.

Based on the foregoing simple and illustrative example, one may note that the optimal locations of internal ringe supports are found at the nodal rings of the corresponding internally unsupported plate and this optimality condition also applies when the plate shape is to be optimized as well.

5. ELASTIC INTERNAL RING SUPPORTS

In the foregoing formulation, it has been assumed that the internal ring supports are rigid and the transverse displacement $w(e_i) = 0$, $i = 1, \ldots, M$. In this section, some remarks will be made on the internal supports being elastic with transverse and rotational stiffnesses k_{wi} and k_{ri}, respectively. By modelling the supports as elastic springs, one can readily apply the formulation to optimal design of circular plates mounted on an elastic cylinder or elastic rings, or similar problems.

For such elastic supports, the strain energy U given in equation (2a) has to be augmented by the strain energy U_s of the elastic springs that modelled the supports. This strain energy of the springs is given by

$$U_s = \sum_{i=1}^{M} \pi e_i [k_{wi} w^2(e_i) + k_{ri} \psi^2(e_i)]. \tag{20}$$

In view of the internal supports being elastic and the continuity of w and ψ over the supports, the optimality conditions given by equations (11), (15) and (16) now become

$$e_i \left\{ -D\left[\left(\frac{d\psi}{dr}\right)^2\Bigg|_{e_i^+} - \left(\frac{d\psi}{dr}\right)^2\Bigg|_{e_i^-} \right] - \kappa^2 Gh \left[\left(\frac{dw}{dr}\right)^2\Bigg|_{e_i^-} - \left(\frac{dw}{dr}\right)^2\Bigg|_{e_i^+} \right] \right\}$$
$$= k_{wi} w^2(e_i) + k_{ri} \psi^2(e_i), \tag{21}$$

TABLE 3

Comparison between the optimal solutions of ($N = 2$) segmented plates with internal ring supports and the corresponding plates without any ring support

		Simply supported edge		Clamped edge	
h_o/R	M	Plate with M internal support	Plate without internal support	Plate with M internal support	Plate without internal support
0·001	1	$\lambda_1 = 1236$ $e_1/R = 0{\cdot}490$ $\tau(0) = 3{\cdot}670$ $\tau(0{\cdot}5) = 0{\cdot}925$ $\tau(1) = 0{\cdot}556$	$\breve{\lambda}_2 = 1236$ $\breve{e}_1/R = 0{\cdot}489$ $\tau(0) = 3{\cdot}669$ $\tau(0{\cdot}5) = 0{\cdot}925$ $\tau(1) = 0{\cdot}556$	$\lambda_1 = 2609$ $e_1/R = 0{\cdot}391$ $\tau(0) = 3{\cdot}737$ $\tau(0{\cdot}5) = 0{\cdot}208$ $\tau(1) = 1{\cdot}403$	$\breve{\lambda}_2 = 2609$ $\breve{e}_1/R = 0{\cdot}392$ $\tau(0) = 3{\cdot}759$ $\tau(0{\cdot}5) = 0{\cdot}215$ $\tau(1) = 1{\cdot}391$
0·3	1	$\lambda_1 = 530{\cdot}9$ $e_1/R = 0{\cdot}451$ $\tau(0) = 3{\cdot}430$ $\tau(0{\cdot}5) = 1{\cdot}011$ $\tau(1) = 0{\cdot}501$	$\breve{\lambda}_2 = 530{\cdot}9$ $\breve{e}_1/R = 0{\cdot}451$ $\tau(0) = 3{\cdot}445$ $\tau(0{\cdot}5) = 1{\cdot}010$ $\tau(1) = 0{\cdot}499$	$\lambda_1 = 839{\cdot}7$ $e_1/R = 0{\cdot}361$ $\tau(0) = 3{\cdot}644$ $\tau(0{\cdot}5) = 0{\cdot}106$ $\tau(1) = 1{\cdot}544$	$\breve{\lambda}_2 = 839{\cdot}7$ $\breve{e}_1/R = 0{\cdot}362$ $\tau(0) = 3{\cdot}652$ $\tau(0{\cdot}5) = 0{\cdot}107$ $\tau(1) = 1{\cdot}541$
0·001	2	$\lambda_1 = 8449$ $e_1/R = 0{\cdot}345$ $e_2/R = 0{\cdot}722$ $\tau(0) = 4{\cdot}961$ $\tau(0{\cdot}5) = 0{\cdot}275$ $\tau(1) = 1{\cdot}078$	$\breve{\lambda}_3 = 8449$ $\breve{e}_1/R = 0{\cdot}345$ $\breve{e}_2/R = 0{\cdot}722$ $\tau(0) = 4{\cdot}969$ $\tau(0{\cdot}5) = 0{\cdot}272$ $\tau(1) = 1{\cdot}080$	$\lambda_1 = 13223$ $e_1/R = 0{\cdot}296$ $e_2/R = 0{\cdot}648$ $\tau(0) = 4{\cdot}423$ $\tau(0{\cdot}5) = 0$ $\tau(1) = 1{\cdot}516$	$\breve{\lambda}_3 = 13255$ $\breve{e}_1/R = 0{\cdot}297$ $\breve{e}_2/R = 0{\cdot}646$ $\tau(0) = 4{\cdot}396$ $\tau(0{\cdot}5) = 0$ $\tau(1) = 1{\cdot}521$
0·3	2	$\lambda_1 = 2387$ $e_1/R = 0{\cdot}259$ $e_2/R = 0{\cdot}717$ $\tau(0) = 5{\cdot}337$ $\tau(0{\cdot}5) = 0{\cdot}000$ $\tau(1) = 1{\cdot}333$	$\breve{\lambda}_3 = 2397$ $\breve{e}_1/R = 0{\cdot}259$ $\breve{e}_2/R = 0{\cdot}716$ $\tau(0) = 5{\cdot}183$ $\tau(0{\cdot}5) = 0{\cdot}000$ $\tau(1) = 1{\cdot}363$	$\lambda_1 = 2527$ $e_1/R = 0{\cdot}256$ $e_2/R = 0{\cdot}686$ $\tau(0) = 6{\cdot}512$ $\tau(0{\cdot}5) = 0{\cdot}000$ $\tau(1) = 1{\cdot}098$	$\breve{\lambda}_3 = 2538$ $\breve{e}_1/R = 0{\cdot}256$ $\breve{e}_2/R = 0{\cdot}681$ $\tau(0) = 6{\cdot}342$ $\tau(0{\cdot}5) = 0{\cdot}000$ $\tau(1) = 1{\cdot}132$

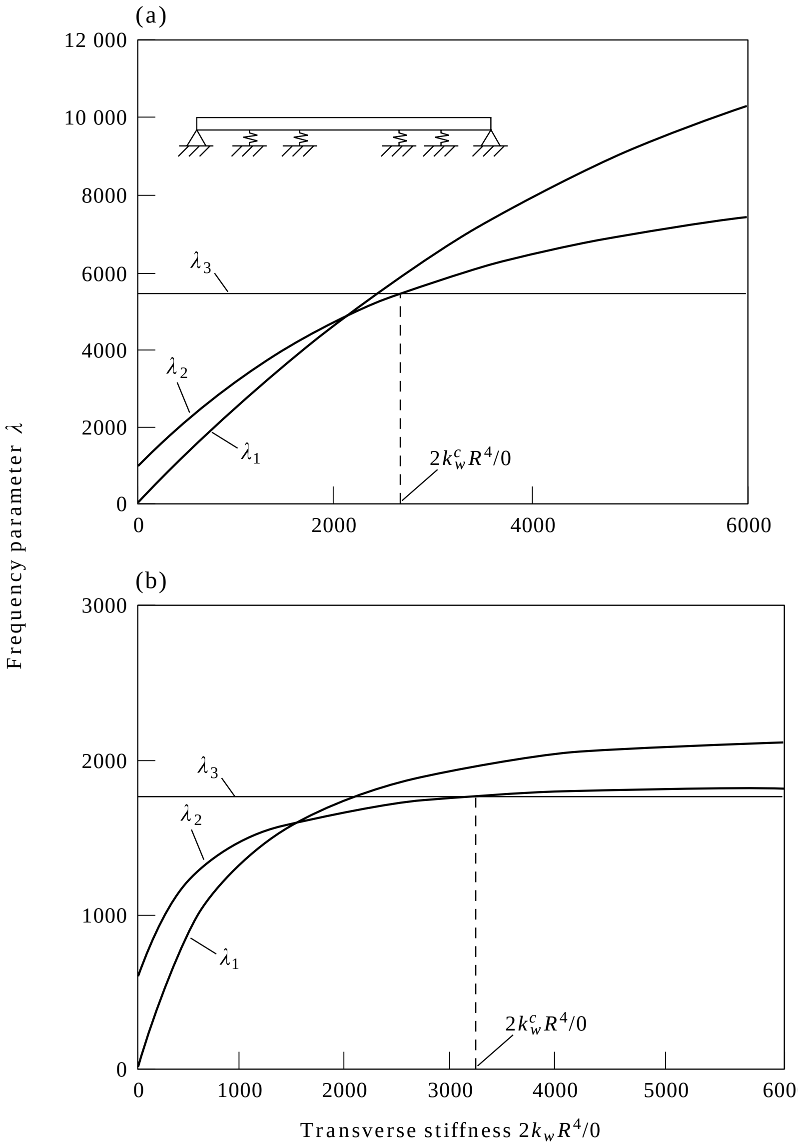

Figure 2. Variations of frequency parameter with respect to transverse stiffness values of two elastic ring supports for a simply supported circular plate with (a) $h/R = 0{\cdot}001$ and (b) $h/R = 0{\cdot}3$.

$$(Q_r)|_{e_i^+} - (Q_r)|_{e_i^-} = P_{wi}|_{e_i} \quad \text{or} \quad \left[\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)\bigg|_{e_i^+} - \left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)\bigg|_{e_i^-}\right] = \frac{k_{wi}}{\kappa^2 Gh}\, w(e_i), \tag{22}$$

$$M_r|_{e_i^+} - M_r|_{e_i^-} = P_{\psi i}|_{e_i} \quad \text{or} \quad \left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)\bigg|_{e_i^+} - \left(\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)\bigg|_{e_i^-}\right] = \frac{k_{ri}}{D}\, \psi(e_i), \tag{23}$$

for $i = 1, 2, \ldots, M$. As expected, equations (22) and (23) show, respectively, that the vertical reaction forces and rotational moments provided by the ith support are proportional to the transverse displacement and rotation angle. Equation (22) also reveals that there is a "kink" in the transverse deflection function w at the elastic support, where the change in slope is proportional to the transverse displacement at that point.

The optimality conditions, given by equations (21) to (23), show that for the case of elastic internal supports with finite stiffnesses, their optimal locations do not match the nodal ring positions of the corresponding plate without internal support. However, if the stiffnesses take on adequate magnitudes so as to cause the transverse displacement and the rotation to be zero at the elastic support locations, then the earlier optimality conditions become valid again. Figures 2(a)

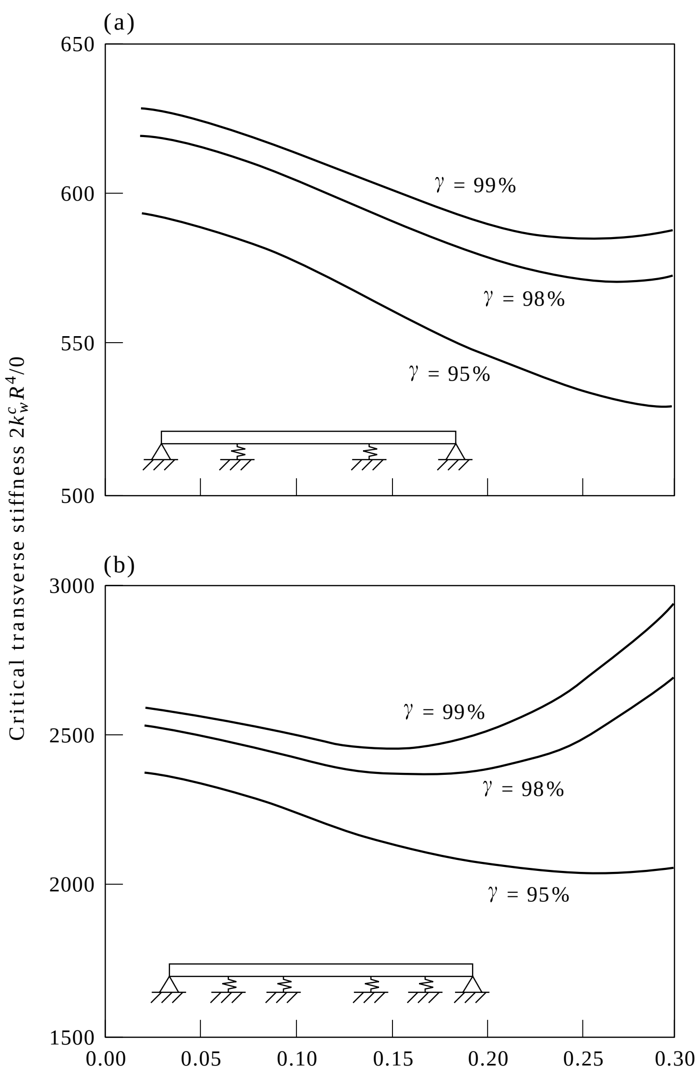

Figure 3. Variations of critical transverse stiffness with respect to h/R for simply supported circular plates in the case of (a) one ring support and (b) two ring supports.

and (b) show the variations in the frequency parameters of simply supported, uniform thickness plates with respect to the transverse stiffness values of the two elastic ring supports placed at $e_1/R = 0{\cdot}279$ and $e_2/R = 0{\cdot}641$. Frequency crossings can be observed from these figures. For low transverse stiffness values, the fundamental frequency parameter is considerably lower than the frequency parameter associated with the third vibration mode. Note that the latter frequency is independent of the transverse stiffness value. As the stiffness value increases, the difference between the first and second frequency values becomes smaller. Eventually the two frequencies are equal when the transverse stiffness reaches a particular value. For this stiffness value, there are two modes of vibration with the same frequency parameter. After the frequency crossing, the former second mode of vibration becomes the fundamental mode of vibration. At the critical stiffness value k_w^c, the fundamental frequency reaches the frequency value associated with the third mode of vibration that in turn corresponds to the fundamental vibration mode of the same plate with two rigid ring supports. As the critical transverse stiffness k_w^c for an internal elastic support (placed at the nodal ring location) is important, plots of these stiffness values for which the fundamental frequency is $\gamma\%$ of the fundamental frequency of the plate with one and two rigid supports are given in Figures 3(a) and (b), respectively. It can be seen that the critical stiffness is affected by the inclusion of transverse shear deformation and rotary inertia as the curves drop with respect to increasing thickness-to-radius and then gradually rise after certain thickness-to-radius values.

6. CONCLUDING REMARKS

It is shown herein that the conditions for optimal locations of rigid, internal ring supports in circular plates for maximum frequency of axisymmetric vibration of a specified order are that the reaction forces and moments must vanish at these locations. The vanishing of these reaction forces and moments imply that the ring supports should be placed at the nodal rings of the appropriate vibration mode of a corresponding plate without any internal ring supports. The optimality condition for the supports is also valid when one considers the more general optimization problem involving the locations of ring supports as well as the plate shape. By knowing the optimality condition, one may bypass the need for optimizing the more difficult plate problem with internal ring supports. Optimization of an internally unsupported plate for the appropriate vibration mode will suffice as the solution will give the optimal support locations from the nodal positions, the optimal plate thickness distribution and the corresponding optimal fundamental frequency value. The latter optimization problem is simpler in the sense that the decision variables have reduced by the number of internal support locations. The optimality conditions also hold in the case of flexible ring supports provided that they have adequate elastic stiffness.

REFERENCES

1. R. COURANT and D. HILBERT 1953 *Methods of Mathematical Physics*, Volume 1. New York: Interscience.

2. N. Olhoff 1976 *Journal of Structural Mechanics* **4**, 122. Optimization of vibrating beams with respect to higher order natural frequencies.
3. G. I. N. Rozvany and Z. Mroz 1977 *Journal of Structural Mechanics* **5**, 279–290. Column design: optimization of support conditions and segmentation.
4. N. Olhoff and J. E. Taylor 1978 *Journal of Structural Mechanics* **6**, 367–382. Designing continuous columns for minimum total cost of material and interior supports.
5. C. M. Wang, K. K. Ang and L. Wang 1995 *Structural Optimization* **9**, 12–17. Optimization of bracing and internal support locations for beams against lateral buckling.
6. D. Szelag and Z. Mroz 1978 *ZAMM* **58**, 501–510. Optimal design of elastic beams with unspecified support conditions.
7. D. Szelag and Z. Mroz 1979 *Computer Methods in Applied Engineering* **19**, 333–349. Optimal design of vibrating beams with unspecified support reactions.
8. Z. Mroz and T. Lekszycki 1981 *Computers and Structures* **14**, 179–185. Optimal support reaction in elastic frame structures.
9. A. Garstecki and Z. Mroz 1987 *Mechanics of Structures and Machines* **15**, 47–68. Optimal design of supports of elastic structures subjected to loads and initial distortions.
10. B. Akesson and N. Olhoff 1988 *Journal of Sound and Vibration* **120**, 457–463. Minimum stiffness of optimally located supports for maximum value of beam frequencies.
11. N. Olhoff and B. Akesson 1991 *Structural Optimization* **3**, 163–175. Minimum stiffness of optimally located supports for maximum value of column buckling loads.
12. R. D. Mindlin 1951 *Transactions of the ASME*, *Journal of Applied Mechanics* **18**, 1031–1036. Influence of rotary inertia and shear in flexural motion of isotropic, elastic plates.
13. F. S. Chou and C. M. Wang 1996 *Structural Optimization* **11**, 128–133. Optimization of linear segmented circular Mindlin plates for maximum fundamental frequency.
14. L. E. Elsgolc 1961 *Calculus of Variations*. Oxford: Pergamon Press.
15. A. W. Leissa 1969 *Vibration of Plates*. NASA SP-160, U.S. Government Printing Office.
16. C. M. Wang 1994 *Transactions of the ASME*, *Journal of Vibration and Acoustics* **116**, 536–540. Natural frequencies formula for simply supported Mindlin plates.
17. C. M. Wang 1996 *Journal of Sound and Vibration* **190**, 255–260. Vibration frequencies of simply supported polygonal sandwich plates via Kirchhoff solutions.